Quantization of Kähler manifolds I: geometric interpretation of Berezin's quantization

J. RAWNSLEY
University of Warwick
M. CAHEN - S. GUTT (*)
Université Libre de Bruxelles

Abstract. *We give a geometric interpretation of Berezin's symbolic calculus on Kähler manifolds in the framework of geometric quantization. Berezin's covariant symbols are defined in terms of coherent states and we study a function* θ *on the manifold which is the central object of the theory. When this function is constant Berezin's quantization rule coincides with the prescription of geometric quantization for the quantizable functions. It is defined on a larger class of functions. We show in the compact homogeneous case how to extend Berezin's procedure to a dense subspace of the algebra of smooth functions.*

0. INTRODUCTION

The aim of this paper is to present Berezin's quantization of Kähler manifolds [2] in a global geometric setting, to compare this procedure with the usual geometric quantization of Kostant and Souriau [4, 8] and to investigate the size of the space of quantizable functions. This paper is a natural extensions of the work of one of us on coherent states [7].

When formulated globally in terms of coherent states, Berezin's construction of covariant symbols of operators appears in terms of sections of the line bundle of geometric

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quantization. The consideration of a system of coherent states leads to the definition of a function θ on the Kähler manifold, which is the central object of the theory. The integral of this function is a topological invariant when the line bundle is sufficiently positive.

A crucial assumption is that θ reduces to a constant; this is the case when the manifold and the quantization are homogeneous. It has been observed in [7] that when θ is constant, Kodaira's holomorphic embedding in a certain complex projective space $P^N(\mathbb{C})$ is symplectic. Furthermore the pull back of the dual of the canonical line bundle on $P^N(\mathbb{C})$ to the manifold is isomorphic to the original quantization bundle. We include the proof of this property for the sake of completeness.

Another very different consequence of the constancy of θ is that Berezin's quantization rule coincides with the rule of geometric quantization for the so-called quantizable. It generalizes geometric quantization in the sense that it applies to a larger class of functions.

Finally we show that when θ is constant and the manifold is compact (in particular for compact homogeneous Kähler manifolds) Berezin's procedure applies to an algebra of functions which is dense in the algebra of smooth functions.

We shall study in part II the relations between Berezin's construction and quantization by deformation [1, 5].

1. GEOMETRIC QUANTIZATION [4, 8]

The geometric quantization of Kostant and Soriau associates a separable Hilbert space $\mathcal{H}$ in a «natural» way to the phase space (= symplectic manifold (M,ω)) of some classical systems with a finite number of degrees of freedom. It singles out a class $\mathcal{E}$ of smooth real valued functions on M (the so called quantizable functions) and maps each element f of $\mathcal{E}$ onto an operator Q_f (the quantum operator corresponding to the classical observable f). We briefly recall this construction as this will give us the opportunity to define our notation.

We shall denote by (M,ω) a smooth (C^∞) connected, symplectic manifold of dimension $m = 2n$. If $\varphi : M \to \mathbf{R}$ is a smooth function, the associated vector field X_φ is defined by:

$$i(X_\varphi)\omega = d\varphi. \tag{1.1}$$

The Poisson bracket $\{\varphi, \psi\}$ of two smooth functions is:

$$\{\varphi, \psi\} = X_\varphi \psi = -\omega(X_\varphi, X_\psi). \tag{1.2}$$

The first element of the construction is a hermitian complex line bundle $\pi : L \to M$ with a connection ∇ leaving the hermitian structure invariant. If h denotes the hermitian structure, if s and s' are smooth sections of L and if X is a vector field on M, the invariance condition reads:

$$X(h(s,s')) = h(\nabla_X s, s') + h(s, \nabla_X s'). \tag{1.3}$$

We shall assume in all that follows that hermitian products are complex linear in their first argument and complex semi-linear in their second argument. The curvature R of (L,∇) is the complex valued 2-form on M such that:

$$(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})s = R(X,Y)s \tag{1.4}$$

where X, Y are smooth vector fields on M and s is a smooth section of L. The triple (L,∇,h) is a *prequantization bundle* over (M,ω) if:

$$\frac{iR}{2\pi} = \omega. \tag{1.5}$$

Such a prequantization bundle exists if and only if ω has integral periods. Clearly if (1.5) is satisfied the curvature form is pure imaginary.

We shall consider the space of smooth sections s of M such that

$$||s||^2 = \int_M h(s,s)\frac{\omega^n}{n!} < \infty \tag{1.6}$$

and denote by $\mathcal{H}(M)$ its L^2-completion.

The second element of the construction is a Hilbert subspace $\mathcal{H}$ of $\mathcal{H}(M)$, the space of polarized sections. We shall assume, from now on, that (M,ω) admits a *positive Kähler polarization* F. That is, there exists on M a smooth complex distribution F, of complex dimension n such that:

(i) $F \cap \bar{F} = 0$;
(ii) $(\bar{F} + F)_x = M_x^{\mathbb{C}}$ (= complexified tangent space at x) for all x in M;
(iii) F is involutive;
(iv) when extended complex linearly, $\omega_{|_F} = 0$;
(v) for all $X \neq 0$ in $F, i\omega(X,\bar{X}) > 0$.

If one defines an almost complex structure J on M, by $J_{|F} = -iI_{|F}$ and $J_{|\bar{F}} = i_{|\bar{F}}$, this almost complex structure is integrable and thus there exists a unique complex analytic structure on M which induces this J. A positive definite metric g on M is defined by:

$$g(X,Y) = \omega(X,JY) \qquad X,Y = \text{vector fields on } M. \tag{1.7}$$

This metric is hermitian, and as ω is closed it is a Kähler metric, i.e. $DJ = 0$ where D is the Levi-Civita connection associated to g. We shall refer to the polarization by F or by J, whichever is the most convenient.

The line bundle $\pi : L \to M$ has then a natural structure of a holomorphic line bundle. A section $s : M \to L$ is said to be holomorphic if $\nabla_X s = 0$, for all X in F. Let $\sigma : U \to L_0$ ($= L\backslash$ zero section) be a smooth section over the open set U of M; let $s : M \to L$ be a smooth section and let $\tilde{s} : U \to \mathbb{C}$ be defined by $s(x) = \sigma(x)\tilde{s}(x)$ for all x in U. The connection 1-form α on L_0 is the unique 1-form such that

$$(1.8) \qquad \widetilde{\nabla_X s}(x) = (X\tilde{s})(x) + (\sigma^*\alpha)_x(X)\tilde{s}(x)$$

for any x in U, any vector field X on U, any section s, any U open in M and any smooth section σ on U with values in L_0. If one chooses locally a holomorphic section $\sigma : U \to L_0$, the pull back of the connection form, $\sigma^*\alpha$, is of type $(1,0)$; as the curvature form R is of type $(1,1)$, $\sigma^*\alpha$ is ∂ closed and by the Dolbeault lemma, $\sigma^*\alpha$ is the ∂-differential of a function. As the hermitian structure h of L is invariant one has:

$$(1.9) \qquad \sigma^*\alpha + \overline{\sigma^*\alpha} = \mathrm{d}(lnh)$$

Hence:

$$(1.10) \qquad \sigma^*\alpha = \partial(lnh),$$

$$(1.11) \qquad \omega = \frac{i}{2}\bar{\partial}\partial(lnh).$$

Here we have also written h for the function on $U, x \to h(\sigma(x), \sigma(x))$. Let $\mathcal{H}$ be the space of holomorphic sections s of L such that $||s||^2 < \infty$ (cf. 1.6); it is complete [4] and hence a Hilbert space.

The third element of the construction is the class $\mathcal{E}$ of quantizable classical observables.

Let us first recall that an automorphism $\tilde{\varphi}$ of the hermitian line bundle with connection (L, h, ∇) is an automorphism of the line bundle L such that

(i) $h(\tilde{\varphi}\xi, \tilde{\varphi}\xi) = h(\xi, \xi) \qquad \forall \xi \in L$

(ii) $(\tilde{\varphi}_{|L_0})^*\alpha = \alpha \qquad \alpha =$ connection 1-form on L_0

We shall denote by φ the diffeomorphism of M induced by $\tilde{\varphi}$ (i.e. $\pi \circ \tilde{\varphi} = \varphi \circ \pi$); it is a symplectic diffeomorphism. Let X be a complete vector field on (M, ω) and let φ_t be the one parametric group of diffeomorphisms it generates; the group φ_t can be lifted to a one-parameter group $\tilde{\varphi}_t$ of automorphisms of (L, h, ∇) if and only if X is the hamiltonian vector field corresponding to a certain function f on M (i.e. $X = X_f$ and $i(X_f)\omega = \mathrm{d}f$). The generator Y of $\tilde{\varphi}_t$ is the vector field

$$(1.12) \qquad Y(\xi) = \bar{X}_f(\xi) + 2\pi i f(\pi(\xi))^*(\xi), \qquad \xi \in L_0,$$

where $\tilde{X}_f$ is the horizontal lift of X and

$$2\pi i f(\pi(\xi))^*(\xi) = \frac{d}{dt}(\xi e^{2\pi i t f(\pi(\xi))})_0.$$

The lift of hamiltonian vector fields determines a linear map

$$C^\infty(M, \mathbf{R}) \to \operatorname{End} \Gamma^\infty(L)$$

$$f \to \left[s \to Q_f s = \nabla_{X_f} s = 2\pi i f s \right] \tag{1.13}$$

It has the property that:

$$\left[Q_f, Q_g \right] = Q_{\{f,g\}}, \qquad f, g \in C^\infty(M, \mathbf{R}). \tag{1.14}$$

In order that Q_f be defined on $\mathcal{H}$ it is necessary that for any holomorphic section $s, \nabla_X Q_F s = 0$ for any X in F. This is equivalent to saying that $[X, X_f]$ belongs to F for any X in F. We thus define $\mathcal{E}$, the *space of quantizable functions*, as:

$$\mathcal{E} = \left\{ f \in C^\infty(M, \mathbf{R}) | [X_f, X] \in F, \quad \forall X \in F \right\}. \tag{1.15}$$

If f and g belong to $\mathcal{E}$, so does their Poisson bracket.

To summarize, a Kähler manifold (M, ω, F) is said to be *quantizable* if there exists over M a hermitian, holomorphic line bundle with connection (L, h, ∇) such that the curvature R is related to the Kähler for ω by

$$R = -2\pi i \omega$$

A choice of such a line bundle is called a *quantization* of the Kähler manifold.

The basic *Hilbert space* of the theory $\mathcal{H}$ is the space of holomorphic sections of L which are L^2 with respect to the Liouville measure on M.

The *space of quantizable functions*, $\mathcal{E}$, is the set of smooth functions whose corresponding hamiltonian vector field stabilize the antiholomorphic subbundle F of the complexified tangent space, $(TM)^{\mathbb{C}}$. If f belongs to $\mathcal{E}$ the *quantum operator* Q_f is the linear operator on $\mathcal{H}, Q_f = \nabla_{X_f} - 2\pi i f$.

REMARK. We have adopted units relative to which the Planck constant $\hbar = 1$; to conform with standard usage we should in fact consider $\frac{\hbar}{i} Q_f$ as the quantum operator.

EXAMPLE. If $(M,\omega) = (\mathbb{R}^{2n}, \sum_{j\leq n} \mathrm{d}x_j \wedge \mathrm{d}y_j)$, the line bundle $L = \mathbb{R}^{2n} \times \mathbb{C}$. Taking complex coordinates $z_j = x_j + iy_j$ and a polarization F spanned by $\left\{\partial_{\bar{z}_j};\ j \leq n\right\}$, one chooses for the connection form an L_0 :

$$\alpha = -\pi \sum_j \bar{z}_j \mathrm{d}z_j + \mathrm{d}u/u \qquad (u = \text{coordinate on } \mathbb{C})$$

The section $\sigma : \mathbb{R}^{2n} \to L_0 : (x,y) \to (x,y;1)$ is holomorphic and the function $h(x,y) \underset{\text{not}}{=} h(\sigma(x,y),\sigma(x,y))$ is given by:

$$h = h_0 e^{-\pi \sum_{j\leq n} z_j\bar{z}_j} \qquad h_0 \in \mathbb{R}_0^+$$

The space $\mathcal{H} = \{f : \mathbb{R}^{2n} \to \mathbb{C} \mid$ holomorphic and such that:

$$\int_{\mathbb{R}^{2n}} (i)^n e^{-\pi \sum_j z_j\bar{z}_j} |f(z)|^2 \sqcap_j \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j < \infty\}.$$

The quantizable functions are of the form

$$\sum_{k,l} \alpha_{kl} z_k \bar{z}_l + \sum_k \beta_k z_k + \bar{\beta}_k \bar{z}_k + \gamma$$

where $\bar{\alpha}_{kl} = \alpha_{lk}$ and $\gamma = \bar{\gamma}$.

2. BEREZIN' S QUANTIZATION AND COHERENT STATES

We present Berezin's quantization procedure in the same framwork as geometric quantization using coherent states; this is a natural extension of [7]. We also show, using a local trivialization, how Berezin's original formulation fits with this geometric presentation. Berezin's quantization has been developed by several authors (see for example [5]); neither Berezin, nor its successors seem to have considered the function θ which plays the crucial rôle both from the point of view of geometry (see theorem 1 of § 3) and from the point of view of analysis (see proposition 2 and theorem of § 4).

Let (L,h,∇) be a quantization of the Kähler manifold (M,ω,F); let s be an element of $\mathcal{H}$ and let q belong to L_0 with $\pi(q) = x$. Evaluation of a section s at x gives a multiple $l_q(s)$ of q and as s is holomorphic, $l_q(s)$ is a linear continuous functional of s.

(2.1) $$s(x) = s(\pi(q)) = l_q(s)q$$

By Riesz's theorem, there exists an element e_q in $\mathcal{H}$ such that:

(2.2) $$l_q(s) = \langle s, e_q \rangle$$

where $\langle\ ,\ \rangle$ is the scalar product in $\mathcal{H}$.

Observe that if c belongs to $\mathbb{C}^*$:

(2.3) $$l_{cq}(s) = c^{-1} l_q(s); \quad e_{cq} = \bar{c}^{-1} e_q$$

The sections e_q will be called the *coherent states.*

Let $\tilde{\varphi}$ be an automorphism of the quantization bundle (L, h, ∇) sucht that the corresponding symplectic diffeomorphism φ of M preserves the polarization F; φ is then holomorphic. The automorphism $\tilde{\varphi}$ acts unitarily on $\mathcal{H}$; since

$$(\tilde{\varphi}s)(x) = \tilde{\varphi}s(\varphi^{-1}x) \quad x \in M;\ s \in \mathcal{H}$$

$$\nabla_{\varphi_* X}\tilde{\varphi}s = \tilde{\varphi}\nabla_X s = 0 \quad x \in F;\ s \in \mathcal{H}$$

Hence $\tilde{\varphi}.s$ is holomorphic.

From this one deduces that:

(2.4) $$\tilde{\varphi}.e_q = e_{\tilde{\varphi}(q)}.$$

Indeed for any s in $\mathcal{H}$:

$$s(\varphi(x)) = \langle s, e_{\tilde{\varphi}(q)}\rangle \tilde{\varphi}(q) = (\tilde{\varphi} \circ \tilde{\varphi}^{-1} s)(\varphi(x)) =$$
$$= \tilde{\varphi}((\tilde{\varphi}^{-1}s)(x)) = \tilde{\varphi}(\langle \tilde{\varphi}^{-1}s, e_q\rangle q) = \langle s, \tilde{\varphi}e_q\rangle \tilde{\varphi}(q)$$

Equation (2.4) means that the quantum evolution of the coherent states follows the classical evolution.

Consider now a bounded operator $A : \mathcal{H} \to \mathcal{H}$. Following Berezin we associate to this operator a covariant symbol $\hat{A}$ which is a complex valued function defined on M by

(2.5) $$\hat{A}(x) = \frac{\langle Ae_q, e_q\rangle}{\|e_q\|^2} \qquad q \in L_0, \pi(q) = x$$

This makes sense because of (2.3). To obtain a quantization we have to reverse this procedure; the class of quantizable functions f on M is the set $\hat{E}(L)$ of covariant symbols of bounded operators $\hat{Q}_f$ on $\mathcal{H}$.

Each such covariant symbol can be analytically continued to the open dense set V in $M \times M$ consisting of points $(x, y$ such that $\langle e_{q'}, e_q\rangle \neq 0$ (where $\pi(q) = x$ and $\pi(q') = y$) holomorphically in x and antiholomorphically in y. The analytic continuation is given by

$$\hat{A}(x,y) = \frac{\langle Ae_{q'}, e_q\rangle}{\langle e_{q'}, e_q\rangle}. \tag{2.6}$$

The operator A can be recovered from its symbol as follows:

$$\begin{aligned} As(x) &= \langle As, e_q\rangle q = \langle s, A^* e_q\rangle q \\ &= \int_M h_y(s(y), (A^* e_q)(y)) \frac{\omega^n}{n!}(y) q \\ &= \int_M h_y(s(y), e_q(y)) \hat{A}(x,y) \frac{\omega^n}{n!}(y) q \end{aligned} \tag{2.7}$$

where $\pi(q) = x$ and $q \in L_0$.

To compare these formulaa with the ones given by Berezin we take a dense open set U in M where there exists a holomorphic section $s_0 : U \to L_0$. Any element s in $\mathcal{H}$, when restricted to U can be expressed

$$s(x) = f(x) s_0(x), \qquad x \in U \tag{2.8}$$

where $f : U \to \mathbb{C}$ is holomorphic. Furthermore the map $s \to f$ is an isometry of $\mathcal{H}$ to a space of holomorphic functions on U which belong to L^2, for the measure $\mu = |s_0|^2 \frac{\omega^n}{n!}$ where $|s_0|^2 = h(s_0, s_0)$. This Hilbert space will be denoted by $\tilde{L}^2_\mu$.

If:

$$e_{s_0}(x) \underset{\text{not}}{=} f_x s_0 \tag{2.9}$$

we can define a map $U \to \tilde{L}^2_\mu : x \to f_x$. The functions f_x are often called coherent states but we shall here reserve this terminology for the e_q's.

If g belongs to $\tilde{L}^2_\mu$, one has:

$$g(x) = \langle g, f_x\rangle_\mu = \int_U g(y) \overline{f_x(y)} |s_0|^2(y) \frac{\omega^n}{n!}(y) \tag{2.10}$$

In particular:

$$\overline{f_x(y)} = \langle f_y, f_x\rangle_\mu \underset{\text{not}}{=} k(x,y) \tag{2.11}$$

This function k, which is defined on $U \times U$, is holomorphic in the first argument and antiholomorphic in the second; it is known as the *reproducing kernel.* The property:

(2.12) $$g(x) = \int_U g(y)\, k(x,y)\, \mu(y)$$

justifies the name. In view of the analyticity properties of k, it is uniquely determined by its restriction to the diagonal.

Consider now a bounded operator A on $\mathcal{H}$ and define the corresponding operator A_0 on $\tilde{L}^2_\mu$ by:

(2.13) $$As = (A_0 f) s_0 \qquad s \in \mathcal{H};\ s = f s_0 \text{ on } U.$$

The analytic continuation of the covariant symbol $\hat{A}$, when restricted to $V \cap (U \times U)$ has the expression

(2.14) $$\hat{A}(x,y) = \frac{\langle A_0 f_y s_0, f_x s_0 \rangle}{\langle f_y s_0, f_x s_0 \rangle} = \frac{\langle A_0 f_y, f_x \rangle_\mu}{k(x,y)}$$

This is the formula introduced by Berezin.

When M is compact the space $\mathcal{H}$ of holomorphic sections of L is finite dimensional; all linear operators on $\mathcal{H}$ are finite rank, bounded, and are linear combinations of operators of rank one. The symbol of such an operator is easily determined; let $u, v \in \mathcal{H}$ and let:

(2.15) $$As = \langle s, u \rangle v, \qquad s \in \mathcal{H}.$$

Its symbol $\hat{A}(x)$ has the expression:

(2.16) $$\hat{A}(x) = \frac{h(v(x), u(x))}{|q|^2 \|e_q\|^2}$$

and one checks that the denominator is a well defined function on M.

3. THE FUNCTION θ

The function θ on M :

(3.1) $$\theta(x) = |q|^2 \|e_q\|^2 \qquad q \in L_0 \text{ and } \pi(q) = x$$

has been introduced in [7] where it was denoted by ϵ and plays a crucial role in this theory.

In terms of a holomorphic local trivialization $s_0 : U \to L_0$ we have:

$$\theta(x) = |s_0(x)|^2 \|e_{s_0(x)}\|^2 \qquad x \in U.$$

In particular when θ is constant

$$\|e_{s_0(x)}\|^2 = \frac{\theta}{|s_0(x)|^2} \tag{3.2}$$

and then the coherent states can be determined by analytical extension. Indeed on U :

$$e_{s_0(x)}(y) = \langle e_{s_0(x)}, e_{s_0(y)}\rangle\, s_0(y)$$

and $\langle e_{s_0(x)}, e_{s_0(y)}\rangle$ is holomorphic in y, antiholomorphic in x on $U \times U$ and thus, as previously stated, uniquely determined by its restriction to the diagonal: $\|e_{s_0(x)}\|^2$.

Assume M is compact and let $\{s_i;\ i \leq N\}$ be an orthonormal basis of $\mathcal{H}$. Then

$$\theta(x) = \sum_{i=1}^{N} h(s_i(x), s_i(x))$$

Indeed if $s \in \mathcal{H}$ and if $q \in L_0$ and $\pi(q) = x$:

$$s(x) = \sum_{i=1}^{N} \langle s, s_i\rangle s_i(x) = \sum_{i=1}^{N} \langle s, s_i\rangle \lambda_i q$$

$$= \langle s, \sum_{i=1}^{N} \bar{\lambda}_i s_i\rangle q$$

if $s_i(x) = \lambda_i q$. Hence $e_q = \sum_{i=1}^{N} \bar{\lambda}_i s_i$ and:

$$\theta(x) = |q|^2 \|e_q\|^2 = |q|^2 \sum_{i=1}^{N} |\lambda_i|^2 = \sum_{i=1}^{N} h(s_i(x), s_i(x))$$

EXAMPLE 1. When M is a homogeneous space, and L a homogeneous line bundle over M, this function θ is a constant. More generally if φ is a symplectic diffeomorphism of (M, ω), stabilizing the polarization F and if $\tilde{\varphi}$ is an automorphism of the quantization bundle which lifts φ, one knows (cf. 2.4) that $\tilde{\varphi} e_q = e_{\tilde{\varphi} q}$: furthermore $\tilde{\varphi}$ acts unitarily on $\mathcal{H}$ and thus $\varphi^* \theta = \theta$.

EXAMPLE 2. It has been shown in [6] that θ is not a constant for the Kähler polarization of the Kepler manifold.

EXAMPLE 3. The complex torus $T = \frac{\mathbb{C}^n}{\Gamma}$ is a compact homogeneous symplectic manifold which is integral if $\rho(\Gamma,\Gamma) \in \mathbb{Z}$ where ρ is the linear symplectic structure on $\mathbb{C}^n$ inducing the symplectic form ω on T. There is an action of Γ on the trivial bundle $\tilde{L}$ on $\mathbb{C}^n$ such that the quotient L is a holomorphic line bundle over T with Chern class $[\omega]$. For this bundle L, θ can not be constant since locally L and $\tilde{L}$ are isomorphic; hence formula (3.2) would imply that they have the same coherent states and those of $\tilde{L}$ are not Γ-periodic.

Assume M compact and let A be a linear operator on $\mathcal{H}$, then if $\{s_i; i \leq N\}$ is an orthonormal basisi of $\mathcal{H}$:

$$
\begin{aligned}
TrA &= \sum_i \langle As_i, s_i\rangle = \sum_i \int_M h((As_i(x), s_i(x))\ \frac{\omega^n}{n!} \\
&= \sum_i \int_M \langle As_i, e_q\rangle h(q, s_i(x))\ \frac{\omega^n}{n!} \\
&= \sum_i \int_M h(q, \langle A^* e_q, s_i\rangle s_i(x))\ \frac{\omega^n}{n!} \\
&= \sum_i \int_M h(q, \langle A^* e_q, s_i\rangle \langle s_i, e_q\rangle q)\ \frac{\omega^n}{n!} \qquad (3.2)\\
&= \int_M h(q, \langle A^* e_q, e_q\rangle q)\ \frac{\omega^n}{n!} \\
&= \int_M |q|^2 \hat{A}(x) \|e_q\|^2\ \frac{\omega^n}{n!} \\
&= \int_M \hat{A}(x)\theta(x)\ \frac{\omega^n}{n!}
\end{aligned}
$$

In particular if $A = I$, one gets:

$$\dim \mathcal{H} = \int_M \theta(x)\ \frac{\omega^n}{n!}. \qquad (3.3)$$

If L is homogeneous one gets:

$$\theta = \dim \mathcal{H}/\mathrm{vol}\, M. \qquad (3.4)$$

From the Riemann-Roch-Hirzebruch formula [3] and Kodaira's vanishing theorem one gets.

THEOREM. *The integral of θ is a topological invariant when the quantization bundle is sufficiently positive.* ∎

The condition that θ is a constant is necessary and sufficient for the geometric quantization construction to be «projectively induced» [7]. We assume (M,ω,F) to be a compact Kähler manifold and we denote by (L,∇,h) a quantization bundle; let $\{s_i; i\leq N\}$ be an orthonormal basis of $\mathcal{H}$ (= space of holomorphic sections of L). We assume that for any x in M there exists a holomorphic section $s(\epsilon\mathcal{H})$ which does not vanish at x.

Let $p:\mathcal{H}^*\backslash\{0\}\to P(\mathcal{H}^*)$ (= projective space of $\mathcal{H}^*$) be the canonical projection; we define Kodaira's map $\varphi: M\to P(\mathcal{H}^*) : x\to p[s\to l_q(s)]$ where $s\in\mathcal{H}, q\in L_x^0$ and $l_q(s)q = s(x)$. A family of charts $V_i : i\leq N)$ on $P(\mathcal{H}^*)$ is characterized by:

$$V_i : \{p(v)\,|v\in\mathcal{H}^*\backslash\{0\} \quad \text{and} \quad v(s_i)\neq 0\}.$$

The local complex coordinates on V_i are:

$$J^j_{(i)}(p(v)) = \frac{v(s_j)}{v(s_i)}, \qquad (j\leq N,\ j\neq i).$$

Introduce the functions

$$l^i_j(x) = l_{s_i(x)}(s_j).$$

These are defined and holomorphic on the open set $U_i = \{x\in M|s_i(x)\neq 0\}$ as $s_j(x) = l^i_j(x)s_i(x)$. If $s=\sum_{k\leq N} c_k s_k$:

$$\varphi(x) = p\left[s\to\sum_{k\leq N} l^i_k(x)c_k\right] = p\left(\sum_{k\leq N} l^i_k(x)s^k_*\right) \qquad x\in U_i$$

where $\{s^i_*;\ i\leq N\}$ is the dual basis of $\mathcal{H}^*$. This shows that φ is a holomorphic map.

Let $a : K^*\to P(\mathcal{H}^*)$ be the dual of the canonical line bundle on $P(\mathcal{H}^*)$; let $K^*_0 = K^*\backslash\{\text{zero section}\}$. The line bundle K^* is associated to the principal $\mathbb{C}^*$ bundle $p:\mathcal{H}^*\backslash\{0\}\to P(\mathcal{H}^*)$ (i.e. $K^* = \mathcal{H}^*\backslash\{0\}\times_\rho\mathbb{C}$ where ρ denotes the action of $\mathbb{C}^*$ on $\mathbb{C}$ defined by $\rho(\mu)z = \mu^{-1}z$).

One identifies K^*_0 with $\mathcal{H}^*\backslash\{0\}$ as follows: an element of K^*_0 is an equivalence class $[u,z]$ (where $u\in\mathcal{H}^*\backslash\{0\}$ and $z\in\mathbb{C}^*$; recall that $[u,z]=[u\mu,\mu z]$ for any $\mu\in\mathbb{C}^*$); the map $\psi : K^*_0\to\mathcal{H}^*\backslash\{0\}$ is given by: $\psi([u,z]) = \frac{u}{z}$. The connection 1-form β on $\mathcal{H}^*\backslash\{0\}$ is defined by:

$$\beta_v(w) = \frac{\langle v,w\rangle}{||v||^2} \qquad v\in\mathcal{H}^*\backslash\{0\}; \qquad w\in\mathcal{H}^*$$

A vector tangent to $\mathcal{H}^*\backslash\{0\}$ is viewed as an element of $\mathcal{H}^*$ and the hermitian structure on $\mathcal{H}^*$ is induced by the Hilbert structure on $\mathcal{H}$ (i.e. if $\alpha:\mathcal{H}\to\mathcal{H}^*$ is the semi linear map $(\alpha(x))(y) = \langle y, x\rangle$ one defines $\langle\alpha(v_1), \alpha(v_2)\rangle = \langle v_1, v_2\rangle$; this scalar product is semi linear in the first factor and linear in the second). The hermitian form h on K^* is defined by:

$$h([u,z],[u,z]) = \frac{z\bar{z}}{\|u\|^2} \qquad u \in \mathcal{H}^*\backslash\{0\}, z \in \mathbb{C}$$

where $\|u\|^2$ denotes the square of the norm in $\mathcal{H}^*\backslash\{0\}$.

This hermitian structure is invariant by the connection. Indeed if $\sigma_i : V_i \to \mathcal{H}^*\backslash\{0\} : p(v) \to \frac{v}{v(s_i)}$ is a local section of $\mathcal{H}^*\backslash\{0\}$ and $\bar{\sigma}_i : V_i \to K^* : p(v) \to [\sigma_i(p(v)), 1]$ is the corresponding section of K^* one sees that:

$$\left(\nabla_{p_* w}\bar{\sigma}_i\right)_{p(v)} = \left(\frac{w(s_i)}{v(s_i)} - \beta_v(w)\right)\bar{\sigma}_i(p(v))$$

and as $h_{p(v)}(\bar{\sigma}_i(p(v)), \bar{\sigma}_i(p(v)) = \frac{|v(s_i)|^2}{\|v\|^2}$, invariance is checked by straight-forward calculation.

Observe that $\bar{\sigma}_i$ is a holomorphic section of K^* and that β is a 1-form of type $(1,0)$ which reads:

$$\beta_v = \partial(\log \|v\|^2)$$

The curvature form Ω of $\mathcal{H}^*\backslash\{0\}$ is of the form:

$$\Omega = -2\pi i p^*\omega_0$$

where ω_0 is a real symplectic 2-form on $P(\mathcal{H}^*)$ invariant by the natural action of the unitary group on projective space.

The above shows that (K^*, ∇, h) is a quantization on the Kähler manifold $(P(\mathcal{H}^*), \omega_0)$

Let $\tilde{\varphi} : L_0 \to K_0^* : q \to l_q$; this map is holomorphic and a homomorphism of principal $\mathbb{C}^*$ bundle; it projects onto the Kodaira map $\varphi : M \to P(\mathcal{H}^*) : x = \pi(q) \to \mathbb{C} l_q$. The map $\psi : K_0^* \to \mathcal{H}^*\backslash\{0\} : [u,z] \to u/z$ projects onto the identity map of $\mathcal{H}^*\backslash\{0\}$. It has the property that $\psi([u,z]c) = \psi([u,z])\,\frac{1}{c}(c \in \mathbb{C}^*)$ and hence if λ^* is a fundamental vector field associated to the action of $\mathbb{C}^*$ on K_0^*, it is mapped by ψ_* on $-\underline{\lambda}^*$ (= fundamental vector field associated to the action of $\mathbb{C}^*$ on $\mathcal{H}^*\backslash\{0\}$. This implies in particular that $-\psi^*\beta$ is a connection 1-form on K_0^*; it is the one inducing ∇.

The 1-form $\alpha + \tilde{\varphi}^*\psi^*\beta$ (where α is the connection 1-form on L_0) is $\mathbb{C}^*$ invariant and vanishes on the fibres of L_0. Hence there exists a complex valued 1-form γ on M such that:

$$\alpha + \tilde{\varphi}^*\psi^*\beta = \pi^*\gamma$$

This form is of type $(1,0)$. Indeed if U is an open set in M and if $s_0 : U \to L_0$ is a holomorphic section, then:

$$\gamma = (\pi \circ s_0)^* \gamma = s_0^* \alpha + (\psi \circ \tilde{\varphi} \circ s_0)^* \beta$$

and the 1-forms on the right hand side are of type $(1,0)$.

Using the formula (1.10) one has:

$$\gamma_x = (s_0^* \alpha)_x + (\psi \circ \tilde{\varphi} \circ s_0)^* \beta_x = (\partial \log |s_0|^2 + \partial \log ||ls_0||^2)_x$$
$$= (\partial \log |s_0|^2 ||ls_0||^2)_x = (\partial \theta)_x$$

If θ is a constant function, this constant is necessarily strictly positive as:

$$\theta(x) = \sum_{i=1}^{N} h(s_i(x), s_i(x))$$

and thus for any x in M, there exists a holomorphic section which does not vanish at x.

Hence the:

PROPOSITION. *Let (L, h, ∇) be a quantization of the compact Kähler manifold (M, ω, F) such that the function θ is constant; let $\mathcal{H}$ be the space of holomorphic sections of L and let $\varphi : M \to P(\mathcal{H}^*)$ be Kodaira's map. Then the pull back of the dual of the canonical bundle over projective spaces, with the usual connection and hermitian structure defines a quantization over M isomorphic to the original one.*

4. GEOMETRIC QUANTIZATION AND BEREZIN'S QUANTIZATION

We compare geometric quantization and Berezin's procedure as formulated in § 2. We show when θ is a constant that the space of symbols contains the space $\mathcal{E}$ of quantizable functions. Furthermore we prove that the space of symbols increases with the Chern class of the bundle when all corresponding θ's are constant and that the limit is dense in the space of smooth functions on M.

PROPOSITION 1. *Let φ be a quantizable function on the Kähler manifold (M, ω, F); let $Q(\varphi)$ be the corresponding quantum operator on $\mathcal{H}$. Then its symbol $\widehat{Q(\varphi)}$ is given by:*

$$\widehat{Q(\varphi)} = X^{\prime}_{\varphi}(\ln \theta) - 2\pi i \varphi$$

where $X^{\prime}_{\varphi}$ is the $\bar{F}$-component of the vector field X_φ. In particular if θ is a constant, the symbol of φ is proportional to φ.

Proof. Let φ be a quantizable function and let X_φ be the corresponding hamiltonian vector field. Write $X_\varphi = X'_\varphi + X''_\varphi$, where X'_φ (resp. X''_φ) belongs to $\bar{F}$ (resp. F); the condition $[X, X_\varphi]$ belongs to F for any X in F is equivalent to saying that X'_φ is a holomorphic vector field.

If $s_0 : U \to L_0$ is a holomorphic section and $f : U \to \mathbb{C}$ is a holomorphic function

$$Q(\varphi_0)f = X_\varphi f + (s_0^*\alpha)(X_\varphi)f - 2\pi i\varphi f =$$
$$= X'_\varphi f + (s_0^*\alpha)(X'_\varphi)f - 2\pi i\varphi 1.$$

As $(Q(\varphi))_0 f$ is holomorphic (φ is a quantizable function) one secs that $s_0^*\alpha(X'_\varphi) - 2\pi i\varphi$ is a holomorphic function. Using the same notation as in (2.9, 2.13, 2.14):

$$\langle (Q(\varphi))_0 f_y, f_x\rangle_\mu = (Q(\varphi_0)f_y(x)$$
$$= (X'_\varphi f_y)(x) + \left[(s_0^*\alpha)(X'_\varphi)(x) - 2\pi i\varphi(x)\right] f_y(x)$$

Observe that $f_y(x)$ is antiholomorphic in y and thus:

$$(X'_\varphi f_y)(x)|_{y=x} = (X'_\varphi g)(x). \qquad g(x) = f_x(x) = k(x,x)$$

Also:

$$f_x(x) = \langle f_x, f_x\rangle_\mu = \langle e_{s_0(x)}, e_{s_0(x)}\rangle = \frac{\theta(x)}{|s_0(x)|^2}.$$

Thus:

$$(X'_\varphi f_x)(x) = \frac{X'_\varphi(\theta)(x)}{|s_0(x)|^2} - \frac{\theta(x)}{|s_0(x)|^4} X'_\varphi(|s_0|^2)$$
$$= \frac{X'_\varphi(\theta)(x)}{|s_0(x)|^2} - \frac{\theta(x)}{|s_0(x)|^2}(\partial ln|s_0|^2)(X'_\varphi)$$
$$= \frac{X'_\varphi(\theta)(x)}{|s_0(x)|^2} - \frac{\theta(x)}{|s_0(x)|^2}(s_0^*\alpha)(X'_\varphi).$$

Substituting:

$$\langle (Q(\varphi))_0 f_x, f_x\rangle_\mu = \frac{X'_\varphi(\theta)}{|s_0(x)|^2} - 2\pi i\frac{\theta(x)}{|s_0(x)|^2}\varphi(x),$$

and

$$\widehat{Q(\varphi)}(x) = \frac{\langle (Q(\varphi))_0 f_x, f_x\rangle_\mu}{k(x,x)} = X'_\varphi(ln\theta) - 2\pi i\varphi(x).$$

■

The space of symbols is a finite dimensional subspace of the space of smooth functions on M (we assume M compact in all that follows).

If (L, ∇, h) is a quantization of the Kähler manifold (M, ω, F), $(L^k = \otimes^k L, \nabla^k, h^k)$ is a quantization of the Kähler manifold $(M, k\omega, F)$ for any integer k. The connection ∇^k is defined by using Leibniz's rule and the hermitian structure h^k is given by:

$$h^k(s_0^k, s_0^k) = |s_0|^{2k} \qquad s_0 = \text{local section of } L.$$

Let us denote by $\mathcal{H}^{(k)}$ the space of holomorphic sections of L^k and by $\hat{E}(L^k)$ the space of symbols; let $\theta^{(k)}$ be the function on M associated to $\mathcal{H}^{(k)}$.

PROPOSITION. *Let* $(L, \tilde{V}, h)$ *be a quantization of the compact Kähler manifold* (M, ω, F). *If all functions* $\theta^{(k)}$ *are constant, then the space* $\hat{E}(L^k)$ *is contained in* $\hat{E}(L^{k+1})$ *for all integers* k.

Proof. The space $\hat{E}(L^k)$ is spanned by symbols of operators of rank 1 on $\mathcal{H}^{(k)}$ thus by

$$\{h^{(k)}(v^{(k)}, u^{(k)})/\theta^{(k)} | v^{(k)}, u^{(k)} \in \mathcal{H}^{(k)}\}.$$

Recall that if $\{s_i; i \leq N\}$ is an orthonormal basis of $\mathcal{H}$:

$$\theta(x) = ||e_q||^2 |q|^2 = \sum_i |s_i(x)|^2.$$

One then observes that:

$$\frac{1}{\theta^{(k)}} h^{(k)}(v^{(k)}, u^{(k)})\theta = \sum_i h^{(k)}(v^{(k)}, u^{(k)}) h(s_i, s_i) \frac{1}{\theta^{(k)}}$$
$$= \sum_i h^{(k+1)}(v^{(k)} \otimes s_i, u^{(k)} \otimes s_i) \frac{1}{\theta^{(k)}}.$$

Thus:

$$\frac{\theta\theta^{(k)}}{\theta^{(k+1)}} \hat{E}(L^k \subset \hat{E}(L^{(k+1)})$$

and hence the conclusion when all $\theta^{(k)}$'s are constant. ∎

In view of this nesting property, it is natural to consider the limit $C_L = \bigcup_{k=1}^{\infty} \hat{E}(L^k)$.

THEOREM. *Let* (L, ∇, h) *be a quantization of the compact Kähler manifold* (M, ω, F) *such that, for each integer,* k, *the function* $\theta^{(k)}$ *is a constant. Then* C_L *is a dense subalgebra of* $C^\infty(M)$.

Proof. The space $\hat{E}(L^k)$ is spanned by functions of the form $h^{(k)}(s,t)$ where s and t belong to $\mathcal{H}^{(k)}$; if $u, v \in \mathcal{H}^{(l)}$ one has:

$$h^{(k)}(s,t)h^{(l)}(u,v) = h^{k+l}(s \otimes u, t \otimes v)$$

and thus:

$$\hat{E}(L^{(k)})\hat{E}(L^l) \subset \hat{E}(L^{k+l})$$

The function 1 is the symbol of the identity operator on $\mathcal{H}$ and clearly C_L is stable by conjugation. Thus by Stone-Weierstrass, the theorem is proven provided the functions in C_L separate points of M.

By Kodaira's embedding theorem, as L is a positive line bundle there exists an integer k such that the map:

$$M \to P(\mathcal{H}^{(k)*}) : x = \pi^{(k)}(q) \to p\left[s \to l_q(s)\right]$$

where $q \in L_0^{(k)}$, $s \in \mathcal{H}^{(k)}$ and $p : \mathcal{H}^{(k)*}\backslash\{0\} \to P(\mathcal{H}^{(k)*})$ is the canonical projection on projective space, is an embedding. Assume $\hat{E}(L^k)$ does not separate points of M; then there exist $x \neq y$ in M such that $A(x) = A(y)$ for any $A \in \hat{E}(L^k)$. Let $A = h^{(k)}(e_q, s)$ where $s \in \mathcal{H}^{(k)}$ and $q \in L_0^{(k)}$. Then

$$h^{(k)}(e_q(x), s(x)) = h^{(k)}(e_q(y), s(y))$$

for any choice of q and s. If q_x (resp. q_y) belongs to L_x^0 (resp. L_y^0)

$$\langle e_q, e_{q_x}\rangle h^{(k)}(q_x, s(x)) = \langle e_q, e_{q_x}\rangle h^{(k)}(q_y, s(y))$$

hence:

$$h^{(k)}(s(x), q_x)e_{q_x} = h^{(k)}(s(y), q_y)e_{q_y}$$

for any choice of s. As one has a projective embedding, there exists s such that $s(x) \neq 0$; but then e_{q_x} is proportional to e_{q_y} and the map into projective space is not injective. Thus we have a contradiction and $\hat{E}(L^k)$ must separate points. ∎

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